Problem Sheet 7

Exercise 7.1

Let $F : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ be continuous and bounded, (ξ_n) a collection of i.i.d $\mathcal{Y}$-valued random variables independent of the $\mathcal{X}$-valued random variable X_0. Define

$$X_{n+1} = F(X_n, \xi_n).$$

Prove that the Markov Process X induces a Feller semigroup.

Exercise 7.2

Define the transition probability $P(x, A) = \int_A \phi(y)dy$ if $x \geq 0$ and $P(x, A) = \int_A \psi(y)dy$ if $x < 0$. Give conditions such that the associated semigroup is Feller. Provide an example where this semigroup is not Feller.

Exercise 7.3

Define the translation semigroup

$$(T_t f)(x) = f(x + t).$$

1. Verify the semigroup property and that T_t is a linear isometry on $C_0(\mathbb{R})$ i.e. $\|T_t\| = 1$.
2. Show that (T_t) is strongly continuous on $C_0(\mathbb{R})$.
3. Show that (T_t) is strongly continuous and $\|T_t\| = 1$ on $L^p(\mathbb{R})$ with $1 \leq p < \infty$.
4. Identify the generator of (T_t).
5. Show that (T_t) is not strongly continuous on $L^\infty(\mathbb{R})$.

Exercise 7.4

Let T_t be a strongly continuous semigroup on a Banach Space E. Prove that there exists constants $M \geq 1$, $k \geq 0$ such that $\|T_t\| \leq Me^{kt}$. *Hint: Recall the Uniform Boundedness Principle, that for a collection of continuous linear operators $S_i : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X}$ is a Banach Space, if for every $x \in \mathcal{X}$ we have that* $\sup_i \|S_i(x)\|_{\mathcal{Y}} < \infty$ *then* $\sup_i \|S_i\| < \infty$ *for the operator norm.*

Exercise 7.5

Let T_t be the semigroup induced by a Markov Process whose transition function is absolutely continuous with respect to the Lebesgue Measure. Prove that T_t is not strongly continuous on $\mathcal{B}_b(\mathbb{R})$.

Exercise 7.6

Define the heat kernel $p : (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty)$ by

$$p_t(x, y) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2t}},$$

the associated transition function

$$P_t(x, A) = \int_A p_t(x, y) dy$$

and furthermore the heat semigroup

$$T_t f(x) = \int_{\mathbb{R}^d} f(y) P_t(x, dy).$$

1. Let W be a d-dimensional Brownian Motion. For $0 < s < t$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$, write down an explicit expression for $\mathbb{E}\left[f(W_t) | W_s = x\right]$.
2. Verify that T_t is Feller.
3. You are given that for $p \geq 1$ every $f \in L^p(\mathbb{R}^d)$ and $t > 0$, $\|T_t f\|_{L^p} \leq \|f\|_{L^p}$ (this is proven by Young's Convolution Inequality). Prove that for $f \in L^p(\mathbb{R}^d)$, then

$$\lim_{t \to 0} \|T_t f - f\|_{L^p} = 0.$$

4. Determine the generator of T_t.